

N.B. : (1) Question No. 1 is compulsory.

(2) Attempt any four questions out of remaining six questions.

1. (a) Separate into real and imaginary parts $\sqrt{i}^{\sqrt{i}}$. 5
(b) Prove that the following series is convergent - 5

$$\frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \dots$$

- (c) If $f(x)$ and $g(x)$ are respectively e^x and e^{-x} , prove that c of Cauchy's mean value theorem is the arithmetic mean of a and b . 5
(d) Solve $x^4 + i = 0$. 5
2. (a) Find the unit normal to the surface $xy^3z^2 = 4$ at $(-1, -1, 2)$. 6
(b) If $y = x^n \log x$, prove that $y_{n+1} = \frac{n!}{x}$. 6
(c) Solve (i) $7 \cosh x + 8 \sinh x = 1$ for real values of x . 4

(ii) $\tanh x = \frac{1}{2}$. 4

3. (a) Examine the function $f(x, y) = y^2 + 4xy + 3x^2 + x^3$ for extreme values. 6
(b) Prove that $\sec^2 x = 1 + x^2 + \frac{2x^4}{3} + \dots$ 6
(c) Show that for real values of a and b , 8

$$e^{2ai \cot^{-1} b} \left[\frac{bi - 1}{bi + 1} \right]^{-a} = 1$$

4. (a) If $f(x, y) = (50 - x^2 - y^2)^{1/2}$, find the approximate value of $[f(3,4) - f(3.1, 3.9)]$ 6
(b) If $x = \cos \theta + i \sin \theta$, $y = \cos \phi + i \sin \phi$ prove that - 6

$$\frac{x - y}{x + y} = i \tan \left(\frac{\theta - \phi}{2} \right)$$

- (c) Prove that $\nabla \cdot \left\{ \frac{f(r)}{r} \vec{r} \right\} = \frac{1}{r^2} \frac{d}{dr} \left[r^2 f(r) \right]$ 8

Hence, or otherwise prove that $\text{div}(r^n \vec{r}) = (n + 3)r^n$.

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5. (a) Evaluate $\lim_{x \rightarrow 0} \log_{\tan x} \tan 2x$.

(b) If $\alpha - i\beta = \frac{1}{a - ib}$, prove that $(\alpha^2 + \beta^2)(a^2 + b^2) = 1$

(c) Verify Euler's Theorem for $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$ and also prove that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$$

6. (a) If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, then prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

(b) If $y = \cos(m \sin^{-1}x)$,

Prove that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} + (m^2 - n^2)y_n = 0$.

(c) If w is a 7th root of unity prove that

$s = 1 + w^n + w^{2n} + w^{3n} + w^{4n} + w^{5n} + w^{6n} = 7$ if n is a multiple of 7 and is equal to zero otherwise.

7. (a) Prove that $\log(1+x) = \frac{x}{1+\theta x}$, where $0 < \theta < 1$ and hence deduce that

$$\frac{x}{1+x} < \log(1+x) < x, \quad x > 0.$$

(b) Prove that if the sum and product of two complex numbers are real then the two numbers must be either real or conjugate.

(c) Prove that $e^{\cos^{-1}x} = e^{\pi/2} \left[1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \dots \right]$