F.E (Sem I) All Branches 19/5/09

Applied Maths 11 am to 2 pm.

Con. 2508–09. **(REVISED COURSE)** **VR–1012**

(3 Hours) **[Total Marks : 100**

N.B. : (1) Question No. **1** is **compulsory**.
(2) Attempt any **four** questions out of remaining **six** questions.
(3) Marks are indicated at the **right**.

1. (a) Find the complex number 'z' if $\arg(z+1) = \dfrac{\pi}{6}$ and $\arg(z-1) = \dfrac{2\pi}{3}$ 20

(b) If z is a real no. then show that –

(i) $\sinh^{-1}(z) = \log\left(z + \sqrt{z^2+1}\right)$

(ii) $\cosh^{-1}(z) = \log\left(z + \sqrt{z^2-1}\right)$

(c) If $z = \tan(y + ax) + (y - ax)^{3/2}$

then show that $\dfrac{\partial^2 z}{\partial x^2} = a^2 \dfrac{\partial^2 z}{\partial y^2}$

(d) If $\dfrac{d\bar{a}}{dt} = \bar{u} \times \bar{a}$ & $\dfrac{d\bar{b}}{dt} = \bar{u} \times \bar{b}$

then P.T. $\dfrac{d}{dt}\left[\bar{a} \times \bar{b}\right] = \bar{u} \times \left(\bar{a} \times \bar{b}\right)$.

2. (a) If $y = \dfrac{1}{1 + x + x^2 + x^3}$, find y_n. 6

(b) A rectangular box with open top has volume V. Find the dimensions of the box requiring least material. 6

(c) If $z = f(x, y)$, $x = e^u \cos v$, $y = e^u \sin v$ 8

P.T. (i) $x\dfrac{\partial z}{\partial v} + y\dfrac{\partial z}{\partial u} = e^{2u}\dfrac{\partial z}{\partial y}$

(ii) $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\dfrac{\partial z}{\partial u}\right)^2 + \left(\dfrac{\partial z}{\partial v}\right)^2\right]$.

3. (a) Find the directional derivative of $\phi = x^4 + y^4 + z^4$ at point A (1, –2, 1) in the direction of AB where B is (2, 6, –1). Also find the maximum directional derivative of ϕ at (1, –2, 1). **6**

(b) Find a, b if $\lim_{x \to 0} \frac{a \sinh x + b \sin x}{x3} = \frac{5}{3}$ **6**

(c) If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$ then P.T. **8**

(i) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$

(ii) $\cos(2\alpha) + \cos(2\beta) + \cos(2\gamma) = 0$

(iii) $\cos(\alpha + \beta) + \cos(\beta + \gamma) + \cos(\gamma + \alpha) = 0$

(iv) $\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$

4. (a) If $z = f(x, y)$, $x = e^{u} + e^{-v}$, $y = e^{-u} - e^{v}$ 6

P.T. $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

(b) If $\text{Cosec}\left(\frac{\pi}{4} + ix\right) = u + iv$ then P.T. 6

$(u^2 + v^2)^2 = 2(u^2 - v^2)$

(c) P.T. $\nabla \times \left(\frac{\overline{a} \times \overline{r}}{r^n}\right) = \frac{(2-n)\overline{a}}{r^n} + \frac{n(\overline{a} \cdot \overline{r})\overline{r}}{r^{n+2}}$ 8

5. (a) Using Maclaurin's Series, 6

Prove that $\log(\sec x) = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \ldots\ldots$

(b) Show that $\frac{b-a}{1+b^2} < \tan^{-1}(b) - \tan^{-1}(a) < \frac{b-a}{1+a^2}$ 6

Hence show that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$

(c) Separate into real and imaginary parts $\tan^{-1}(\cos\theta + i\sin\theta)$. 8

6. (a) Test the convergence of – 6

$\frac{x}{1\cdot 2} + \frac{x^2}{3\cdot 4} + \frac{x^3}{5\cdot 6} + \frac{x^4}{7\cdot 8} + \ldots\ldots\ldots (x > 0 \;\&\; x \neq 1)$

(b) If $u = \frac{x^3y^3z^3}{x^3+y^3+z^3} + \log\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$ 6

then P.T. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 6u$.

(c) If $y = 2^x \cos^9 x$, then find y_n. 8

7. (a) If α, β are the roots of the equations $z^2 \sin^2(\theta) - Z\sin(\theta) + 1 = 0$ then prove that – 6

(i) $\alpha^2 + \beta^n = 2\cos(n\theta)\ \ \text{Cosec}^n(\theta)$

(ii) $\alpha^n \cdot \beta^n = \text{Cosec}^{2n}(\theta)$.

(b) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then prove that – 6

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\sin(u)\cos(3u)$

(c) If $z = x\log(x + r) - r$ where $r^2 = x^2 + y^2$ 8

Prove that – (i) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{x+r}$

(ii) $\frac{\partial^3 z}{\partial x^3} = -\left(\frac{x}{r^3}\right)$.